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Convexity of Network Restricted Games Induced by Minimum Partitions

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Abstract

We consider restricted games on weighted communication graphs associated with minimum partitions. We replace in the classical definition of Myerson's graph-restricted game the connected components of any subgraph by the sub-components corresponding to a minimum partition. This minimum partition $\mathcal{P}_{\min}$ is induced by the deletion of the minimum weight edges. We provide necessary conditions on the graph edge-weights to have inheritance of convexity from the underlying game to the restricted game associated with $\mathcal{P}_{\min}$. Then we establish that these conditions are also sufficient for a weaker condition, called $\mathcal{F}$-convexity, obtained by restriction of convexity to connected subsets. Moreover we show that Myerson's game associated to a given graph G can be obtained as a particular case of the $\mathcal{P}_{\min}$-restricted game for a specific weighted graph $G^{'}$. Then we prove that G is cycle-complete if and only if a specific condition on adjacent cycles is satisfied on $G^{'}$.

Keywords: communication networks, cooperative game, restricted game, partitions.

1 Introduction

Communication games were introduced by Myerson in 1977 [11]. These are cooperative games (N, v) defined on the set of vertices N of an undirected graph $G = (N, E)$, where E is the set of edges. v is the characteristic function of the game, $v : 2^N \to \mathbb{R}$, $A \mapsto v(A)$ and satisfies $v(\emptyset) = 0$. The graph G describes how the players of N can communicate: $e = \{i, j\} \in E$ if and only if the players i and j can directly communicate. For every coalition $A \subseteq N$, we consider the induced graph $G_A := (A, E(A))$, where $E(A)$ is the set of edges $e = \{i, j\} \in E$ such that i and j are in A. We denote by A/G the set of connected components of G_A, that is, those sets F which are maximal subcoalitions of A such that all pairs of players i, j in F can communicate

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by a path in G_A linking i and j. Myerson defined the graph-restricted game (N, v^M),

$$v^M(A) = \sum_{F \in A/G} v(F), \text{ for all } A \subseteq N. \tag{1}$$

The new game (N, v^M) takes into account how the players of N can communicate according to the graph G. The inheritance of some important properties as superadditivity and convexity from the underlying game (N, v) to the graph-restricted game (N, v^M) has been investigated. Owen [12] proved that if (N, v) is superadditive then (N, v^M) is also superadditive without any assumption on G. Van den Nouweland and Borm [14] showed that if a communication graph is *cycle-complete*[1] and the underlying game is convex then the graph-restricted game is also convex. Other graph restricted games have been defined when there exists a hierarchical structure on the set of players. Games with permission structures have been introduced in [9] to describe situations where some players need permission of some other players to cooperate within a coalition. They were later extended to games on antimatroids [4]. Other combinatorial structures have been proposed to describe situations which cannot necessarily be represented by a graph [7, 3, 2, 5]. In particular the result on inheritance of convexity by Van den Nouweland and Borm for Myerson's restricted game can also be seen as a consequence of a result by Faigle [7] as proved in [2]. Superadditivity and convexity are important properties in cooperative game theory, since they imply the nonemptiness of the set of imputations, of the core, and that the Shapley value lies in the core.

Myerson's restricted game only takes into account the connectivity between players. It makes sense for a physical communication network as a group of players should be at least able to communicate together to cooperate and get their initial value but it can seem optimistic for social networks. For example let us consider a coalition corresponding to a path as represented in Figure 1. Then this coalition consisting of two terminal players and some intermediaries is supposed to get its initial value as the corresponding subgraph is connected. The coordination of players may be acceptable for

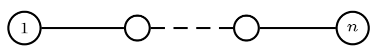

Figure 1: A coalition of players corresponding to a path.

a small number of intermediairies but seems rather unlikely if this number is important. If the group is unable to coordinate then subgroups of players

[1] A graph $G = (N, E)$ is *cycle-complete* if for any cycle $C = (v_1, e_1, v_2, e_2, \ldots, e_m, v_1)$ in G the subset $\{v_1, v_2, \ldots, v_m\} \subseteq N$ of vertices of C induces a complete subgraph in G.

may form and could be reduced to singletons in the worst case. Hence, noting that A/G is a partition of A, a more general setting has been introduced in [10]. We call *correspondence* any way of partitioning, formally a mapping $\mathcal{P}$ on 2^N, assigning to any nonempty $A \subseteq N$ a partition $\mathcal{P}(A)$ of A. Then the *partition restricted game* $(N, \overline{v})$ *associated with* $\mathcal{P}$ is defined by:

$$(2) \qquad \overline{v}(A) = \sum_{F \in \mathcal{P}(A)} v(F), \text{ for all } A \subseteq N.$$

We will more simply refer to this game as the $\mathcal{P}$-restricted game. Let us note that if $\mathcal{P}$ corresponds to the partition into connected components then $(N, \overline{v}) = (N, v^M)$. Then a natural problem is to find necessary and sufficient conditions on correspondences to have inheritance of superadditivity and convexity from (N, v) to $(N, \overline{v})$. A characterization of inheritance of superadditivity and another one of inheritance of convexity but restricted to the family of unanimity games have been established in [10]. For a given correspondence these characterizations imply strong conditions on graphs (as for Myerson game). Therefore it is natural to look for necessary and sufficient conditions on graphs to have inheritance of superadditivity or convexity. Let us observe that even if players are connected they do not necessarily have the same levels of relationships or communication possibilities. Therefore we consider weighted graphs, where each edge $e \in E$ has a weight $w(e)$, whose interpretation may depend on the context (e.g., a degree of friendship, a level of communication, a resistance under attacks, or a security level, etc.). Then, an obvious way of partitioning a coalition $A \subseteq N$, which we denote by $\mathcal{P}_{\min}$, is to remove all edges of minimum weight in A. The reason to do this is that these edges are weak in some sense, and may easily disappear. Of course the weakness of an edge is relative to the subgraph G_A it belongs to. The minimum edge-weight in G_A can be higher than the minimum edge-weight in G but it corresponds to the weakest links between players in coalition A. The components of $\mathcal{P}_{\min}(A)$ should then show the "strongest" components of G_A. In this framework, we study under which necessary conditions the convexity is inherited from (N, v) to $(N, \overline{v})$. The particular case of cycle-free graphs has already been investigated in [10]. Two necessary conditions of convexity on edge-weights were established, the first one on paths and the second one on stars in the graph. It was also observed that for arbitrary graphs a condition on edge-weights of cycles was required. Following the line started in [10] we investigate in this paper the more intricate situation of arbitrary graphs for $\mathcal{P}_{\min}$. We provide two supplementary necessary conditions on edge-weights associated with *pans*[2] and adjacent cycles. The three conditions of [10] are also extended to a more general setting. In particular, paths, stars, and cycles do not necessarily correspond to induced subgraphs. To establish these five necessary conditions we only need to consider con-

[2]A pan graph is a graph obtained by joining a cycle to a vertex by an edge.

nected subsets. Hence these necessary conditions are valid if we only assume $\mathcal{F}$-convexity which is a weaker condition than convexity introduced in [10] and obtained by restricting the definition of convexity to connected subsets. Then we prove the sufficiency of these five conditions for the inheritance of $\mathcal{F}$-convexity from (N, v) to $(N, \overline{v})$ for arbitrary graphs, and this constitutes the main result of the paper (Theorem 20). Of course $\mathcal{F}$-convexity does not imply convexity but we think this notion is of interest in the context of communication networks. Convexity corresponds to the idea that tendency for a player to join a coalition increases as the coalition grows. But if the coalition does not correspond to a connected subset in G then this property seems rather unlikely. Indeed the player will not be able to communicate with the whole group of players in the coalition and in the worst case could be not even linked to any player in the coalition. If we consider that players should be at least able to communicate to cooperate then it becomes natural to restrict convexity to connected subsets. Moreover we give in Section 3 a very simple counter-example to inheritance of convexity with only two different edge-weights for which nevertheless inheritance of $\mathcal{F}$-convexity is satisfied. Hence inheritance of convexity can only occur for a very small class of weighted graphs and it is interesting to consider $\mathcal{F}$-convexity to get inheritance for a sufficiently large class of weighted graphs.

The article is organized as follows. In Section 2 we give preliminary definitions and results established in [10]. In particular, we recall the definition of $\mathcal{F}$-convexity and general conditions on a correspondence to have inheritance of convexity and $\mathcal{F}$-convexity. The $\mathcal{P}_{\min}$-restricted game is defined in Section 3. In Section 4 we first show that Myerson's game associated to a given graph G corresponds to a restriction of the $\mathcal{P}_{\min}$-restricted game associated to a specific weighted graph G' built from G. Then we prove that inheritance of convexity for Myerson's game is equivalent to inheritance of $\mathcal{F}$-convexity for the $\mathcal{P}_{\min}$-restricted game. Section 5 includes the main results ot the paper. In Section 5.1 we establish necessary conditions on edge-weights to have inheritance of $\mathcal{F}$-convexity. Then we prove that these conditions are also sufficient for superadditive games in Section 5.2. As a consequence one only needs to verify inheritance of $\mathcal{F}$-convexity for unanimity games. That is an interesting and non trivial result as a convex game is not in general a convex combination of unanimity games. In Section 5.3 we prove that cycle-completeness of a graph is satisfied if and only if one of the necessary conditions on adjacent cycles is satisfied on the specific graph described in Section 4. Hence this condition on adjacent cycles is particularly relevant. In Section 5.4 we give a description of graphs satisfying the necessary and sufficient conditions for inheritance of $\mathcal{F}$-convexity. This class is large as the number of different edge-weights and of adjacent cycles is not restricted. We conclude with some remarks and suggestions for generalization of these results to other correspondences in Section 6.

2 Preliminary definitions and results

A game (N, v) is *zero-normalized* if $v(i) = 0$ for all $i \in N$. We recall that a game (N, v) is *superadditive* if, for all $A, B \in 2^N$ such that $A \cap B = \emptyset$, $v(A \cup B) \geq v(A) + v(B)$. For any given subset $\emptyset \neq S \subseteq N$, the unanimity game (N, u_S) is defined by:

$$u_S(A) = \begin{cases} 1 & \text{if } A \supseteq S, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

We note that u_S is superadditive for all $S \neq \emptyset$. The following result established in [10] gives general conditions on a correspondence $\mathcal{P}$ to have inheritance of superadditivity.

Theorem 1. *Let N be an arbitrary set and $\mathcal{P}$ a correspondence on N. Then the following claims are equivalent:*

1) For all $\emptyset \neq S \subseteq N$, the $\mathcal{P}$-restricted game $(N, \overline{u_S})$ is superadditive.

2) For all subsets $A \subseteq B \subseteq N$, $\mathcal{P}(A)$ is a refinement of the restriction of $\mathcal{P}(B)$ to A.

3) For all superadditive game (N, v) the $\mathcal{P}$-restricted game $(N, \overline{v})$ is superadditive.

Let $\mathcal{F}$ be a *weakly union-closed family*[3] of subsets of N such that $\emptyset \notin \mathcal{F}$. A game v on 2^N is said to be *$\mathcal{F}$-convex* if for all $A, B \in \mathcal{F}$ such that $A \cap B \in \mathcal{F}$, we have:

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B). \tag{4}$$

We note that u_S is convex for all $S \neq \emptyset$. Of course convexity implies $\mathcal{F}$-convexity which implies also the following condition. If a game v on 2^N is $\mathcal{F}$-convex then, for all $i \in N$ and all $A \subseteq B \subseteq N \setminus \{i\}$ such that A, B and $A \cup \{i\} \in \mathcal{F}$ we have:

$$v(B \cup \{i\}) - v(B) \geq v(A \cup \{i\}) - v(A). \tag{5}$$

Of course if $\mathcal{F} = 2^N \setminus \emptyset$ then $\mathcal{F}$-convexity corresponds to convexity and it is well known that there is equivalence of the two previous conditions [13]. We say that a subset $A \subseteq N$ is connected if the induced graph $G_A = (A, E(A))$ is connected. The family of connected subsets of N is obviously weakly union-closed [3]. For this last family the two previous conditions are also equivalent [10].

[3] $\mathcal{F}$ is weakly union-closed if $A \cup B \in \mathcal{F}$ for all A, $B \in \mathcal{F}$ such that $A \cap B \neq \emptyset$ [8]. Weakly union-closed families were introduced and analysed in [1, 3] and called union stable systems.

Theorem 2. *Let $G = (N, E)$ be an arbitrary graph and let $\mathcal{F}$ be the family of connected subsets of N. Then the following conditions are equivalent:*

$$v \text{ is } \mathcal{F}\text{-convex.} \tag{6}$$

$$\begin{aligned} &v(B \cup \{i\}) - v(B) \geq v(A \cup \{i\}) - v(A), \text{ for all } i \in N, \\ &\text{for all } A \subseteq B \subseteq N \setminus \{i\} \text{ with } A, B, \text{ and } A \cup \{i\} \in \mathcal{F}. \end{aligned} \tag{7}$$

The next theorem established in [10] gives general abstract conditions on a correspondence $\mathcal{P}$ to have inheritance of convexity for unanimity games.

Theorem 3. *Let N be an arbitrary set, $\mathcal{P}$ a correspondence on N, and $\mathcal{F}$ a weakly-union closed family of subsets of N such that $\emptyset \notin \mathcal{F}$. If for all non-empty subset $S \subseteq N$, $(N, \overline{u_S})$ is superadditive, then the following claims are equivalent.*

1) For all non-empty subset $S \subseteq N$, the game $(N, \overline{u_S})$ is $\mathcal{F}$-convex.

2) For all $A, B \in \mathcal{F}$ such that $A \cap B \in \mathcal{F}$, $\mathcal{P}(A \cap B) = \{A_j \cap B_k\,;\, A_j \in \mathcal{P}(A), B_k \in \mathcal{P}(B), A_j \cap B_k \neq \emptyset\}$.

Moreover if $\mathcal{F} = 2^N \setminus \{\emptyset\}$ or if $\mathcal{F}$ corresponds to the set of all connected subsets of a graph then 1) and 2) are equivalent to:

3) For all $i \in N$ and for all $A \subseteq B \subseteq N \setminus \{i\}$ with $A, B,$ and $A \cup \{i\} \in \mathcal{F}$, we have for all $A' \in \mathcal{P}(A \cup \{i\})$, $\mathcal{P}(A)_{|A'} = \mathcal{P}(B)_{|A'}$.

Remark 1. We will mostly use claim 3 in subsequent proofs.

3 $\mathcal{P}_{\min}$-restricted game

Let $G = (N, E, w)$ be an arbitrary graph with an edge-weight function w which assigns a weight $w(e)$ to each edge $e \in E$. We assume that the edge-weights are not all equal to avoid trivial situations. For any subset $A \subseteq N$, we denote by $\sigma(A)$ the minimum edge-weight in G_A, *i.e.*, $\sigma(A) = \min_{e \in E(A)} w(e)$, and by $\Sigma(A)$ the subset of edges of minimum weight in $E(A)$. Let $\mathcal{P}_{\min}$ be the correspondence which associates to every subset $A \subseteq N$ the partition $\mathcal{P}_{\min}(A)$, the elements of which are the components of the graph $G_A = (A, E(A) \setminus \Sigma(A))$. We set $\mathcal{P}_{\min}(\emptyset) = \{\emptyset\}$. Then for every game (N, v) the $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$ is defined by:

$$\overline{v}(A) = \sum_{F \in \mathcal{P}_{\min}(A)} v(F), \text{ for all } A \subseteq N.$$

This game can be considered as a pessimistic alternative to Myerson's game. Indeed to get its initial value a coalition has to be connected but the induced

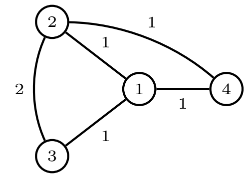

Figure 2: A weighted graph.

subgraph has also to stay connected if we delete the minimum weighted edges. For example, let us consider the weighted graph G represented in Figure 2. If $A = \{1, 2, 3\}$ then $\overline{v}(A) = v(\{1\}) + v(\{2, 3\})$, and if $A = \{1, 2, 4\}$ then $\overline{v}(A) = v(\{1\}) + v(\{2\}) + v(\{4\})$. This basic example shows the specificity of the game associated with $\mathcal{P}_{\min}$. In the two cases the three players can perfectly communicate as they form a complete subgraph. In the first case the edge-weights imply that players 2 and 3 form a subcoalition and will cooperate. But in the second case the three players have the same level of communication. Therefore no particular group will emerge. Hence the $\mathcal{P}_{\min}$-restricted game is very different from Myerson's restricted game. We show in Section 4 that they are actually deeply linked and that the $\mathcal{P}_{\min}$-restricted game is more general as it contains Myerson's game as a specific subcase.

We can immediately observe that inheritance of superadditivity from the underlying game to the $\mathcal{P}_{\min}$-restricted game is always satisfied. Indeed, as for all $A \subseteq B \subseteq N$, $\mathcal{P}_{\min}(A)$ is a refinement of $\mathcal{P}_{\min}(B)_{|A}$, Theorem 1 implies the following result.

Corollary 4. *Let $G = (N, E, w)$ be an arbitrary weighted graph. Then for every superadditive game (N, v), the $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$ is superadditive.*

In comparison inheritance of convexity is very intricate. As pointed out in the introduction we will restrict the study in this paper to inheritance of $\mathcal{F}$-convexity. We end this section with a counterexample to inheritance of convexity. Let us consider the graph represented in Figure 3. We suppose:

$$w_1 = w_2 < w_3 = w_4. \tag{8}$$

We consider $i = 1$, $A_1 = \{2\}$, $A_2 = \{4, 5\}$, $A = A_1 \cup A_2$, and $B = A \cup \{3\}$. Then $\overline{v}(B \cup \{i\}) - \overline{v}(B) = v(i)$ and $\overline{v}(A \cup \{i\}) - \overline{v}(A) = v(A_2) + v(i) - v(4) - v(5)$. Taking $v = u_{A_2}$ we get $\overline{v}(B \cup \{i\}) - \overline{v}(B) = 0 < 1 = \overline{v}(A \cup \{i\}) - \overline{v}(A)$. Therefore there is no inheritance of convexity. It can easily be checked that if A is replaced by any connected subset of B then $\overline{v}(B \cup \{i\}) - \overline{v}(B) = v(i) = \overline{v}(A \cup \{i\}) - \overline{v}(A)$.

Remark 2. We could also use Theorem 3 to prove that there is no inheritance of convexity in the preceding example.

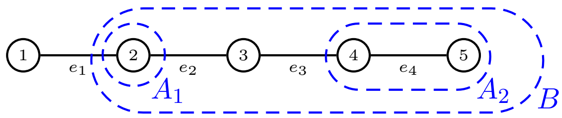

Figure 3: A counterexample to inheritance of convexity.

4 Myerson's game as a particular case of the $\mathcal{P}_{\min}$-restricted game

Given a graph $G = (N, E)$ we build a weighted graph $G^{'} = (N^{'}, E^{'}, w)$ as follows. We add a new vertex s to G and an edge joining s to i for each $i \in N$. Hence $N^{'} = N \cup \{s\}$ and $E^{'} = E \cup \{\{s, i\},\ i \in N\}$. For the weight function, we set $w(e) = 1$ if $e \in E$ and $w(e) = \frac{1}{2}$ if $e \in E^{'} \setminus E$ (we can take any value in $]0, 1[$ instead of $\frac{1}{2}$). For example if G corresponds to a cycle on five vertices then $G^{'}$ corresponds to the graph represented in Figure 4.

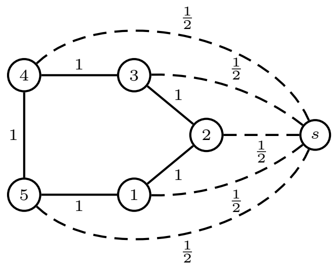

Figure 4: Graph $G^{'}$.

We consider on $G^{'}$ the correspondence $\mathcal{P}_{\min}$. We denote by $\mathcal{P}_M$ the correspondence on G which assigns to each subset $A \subseteq N$ the partition $\mathcal{P}_M(A)$ of A into its connected components. Let us consider a subset $A^{'}$ in $N^{'}$. If $A^{'} \subseteq N$ then $\mathcal{P}_{\min}(A^{'})$ is the singleton partition of $A^{'}$. Otherwise $A^{'} = A \cup \{s\}$ with $A \subseteq N$ and $\mathcal{P}_{\min}(A^{'}) = \{\mathcal{P}_M(A), \{s\}\}$.

Let $(N^{'}, v)$ be a zero-normalized game on $N^{'}$. Then for any $A \subseteq N$ we have :

$$\overline{v}(A) = 0, \tag{9}$$

and setting $\mathcal{P}_M(A) = \{A_1, A_2, \ldots, A_k\}$,

$$\overline{v}(A \cup \{s\}) = \sum_{i=1}^{k} v(A_i) = v^M(A). \tag{10}$$

Let us observe that by (9) and (10) the restricted game $(N^{'}, \overline{v})$ only depends on the restriction of v to N and the restriction of $(N^{'}, \overline{v})$ to the set of coalitions containing s corresponds to the Myerson game on N.

Theorem 5. *Let $G = (N, E)$ be a graph and $G' = (N', E', w)$ be the weighted connected graph obtained from G by the preceding procedure. Let us consider a zero-normalized game (N', v). Then the $\mathcal{P}_{\min}$-restricted game $(N', \overline{v})$ is $\mathcal{F}$-convex if and only if (N, v^M) is convex.*

Proof. Let us assume that $(N', \overline{v})$ is $\mathcal{F}$-convex. Then for any subsets $A, B \subseteq N$, we have:

$$\overline{v}(A \cup B \cup \{s\}) + \overline{v}(A \cap B \cup \{s\}) \geq \overline{v}(A \cup \{s\}) + \overline{v}(B \cup \{s\}). \tag{11}$$

(10) implies that (11) is equivalent to:

$$v^M(A \cup B) + v^M(A \cap B) \geq v^M(A) + v^M(B). \tag{12}$$

Hence $\mathcal{F}$-convexity of $(N', \overline{v})$ implies convexity of (N, v^M).
Let us now assume that (N, v^M) is convex. Then (12) and therefore (11) are satisfied for any subsets $A, B \subseteq N$. As (N, v^M) is convex it is also superadditive and therefore for any subsets $A, B \subseteq N$, we have:

$$v^M(A \cup B) \geq v^M(A), \tag{13}$$

which is equivalent to:

$$\overline{v}(A \cup B \cup \{s\}) + \overline{v}(A \cap B) \geq \overline{v}(A \cup \{s\}) + \overline{v}(B). \tag{14}$$

By symmetry we also have $\overline{v}(A \cup B \cup \{s\}) + \overline{v}(A \cap B) \geq \overline{v}(A) + \overline{v}(B \cup \{s\})$. And $\overline{v}(A \cup B) + \overline{v}(A \cap B) \geq \overline{v}(A) + \overline{v}(B)$ is obviously satisfied for all $A, B \subseteq N'$. Hence $(N', \overline{v})$ is convex. □

Corollary 6. *Let $G = (N, E)$ be a graph and $G' = (N', E', w)$ be the weighted connected graph obtained from G by the preceding procedure. Let us consider a zero-normalized game (N', v). Inheritance of $\mathcal{F}$-convexity from (N', v) to the $\mathcal{P}_{\min}$-restricted game $(N', \overline{v})$ is equivalent to inheritance of convexity from (N, v) to (N, v^M).*

5 Inheritance of $\mathcal{F}$-convexity with $\mathcal{P}_{\min}$

Let $G = (N, E, w)$ be a weighted graph. We denote by w_j the weight of an edge e_j in E. We establish in this part necessary and sufficient conditions on the weight vector w for the inheritance of $\mathcal{F}$-convexity from the original communication game (N, v) to the $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$.

5.1 Necessary conditions on edge-weights

We first establish that there exists a necessary condition associated with every subgraph of G corresponding to a star. A star S_k corresponds to a tree with one internal vertex and k leaves. We present the result for stars with three leaves. The generalization to stars of greater size is immediate. We consider a star S_3 with vertices $\{1,2,3,4\}$ and edges $e_1 = \{1,2\}$, $e_2 = \{1,3\}$ and $e_3 = \{1,4\}$. The following necessary condition already appeared in [10] but was limited to induced stars. It can be extended to all stars of a given graph.

Star Condition. *For every star of type S_3 of G, the edge-weights w_1, w_2, w_3 satisfy, after renumbering the edges if necessary:*

$$w_1 \leq w_2 = w_3.$$

Proposition 7. *Let $G = (N, E, w)$ be a weighted graph, and $\mathcal{F}$ the family of connected subsets of N. If for every unanimity game (N, u_S), the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is $\mathcal{F}$-convex, then the Star Condition is satisfied.*

Proof. We have to prove that we cannot have two edge-weights strictly smaller than a third one. By contradiction let us assume we have $w_1 \leq w_2 < w_3$, after renumbering if necessary. Let us consider the situation of

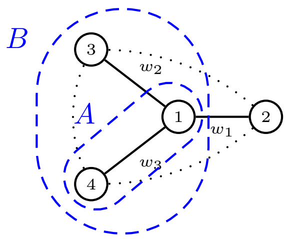

Figure 5: Star of type S_3 with $w_1 \leq w_2 < w_3$.

Figure 5 where $A = \{1,4\}$, $B = \{1,3,4\}$, and $i = 2$. Let us observe that, as the star is not necessarily induced, edges $\{2,3\}$, $\{3,4\}$, and $\{2,4\}$ may exist. By deleting the edges of minimum weight we obtain $\mathcal{P}_{\min}(A) = \{\{1\},\{4\}\}$, and $\mathcal{P}_{\min}(B) = \{A, \{3\}\}$ or $\mathcal{P}_{\min}(B) = \{B\}$. If $\mathcal{P}_{\min}(A \cup \{i\}) = \{A, \{i\}\}$ we have $A \in \mathcal{P}_{\min}(A \cup \{i\})$ but $\mathcal{P}_{\min}(B)_{|A} = A \neq \mathcal{P}_{\min}(A)_{|A}$ and it contradicts Theorem 3. Now if $\mathcal{P}_{\min}(A \cup \{i\}) = \{A \cup \{i\}\}$ we still get a contradiction since we have $\mathcal{P}_{\min}(B)_{|A\cup\{i\}} = A \neq \mathcal{P}_{\min}(A)_{|A\cup\{i\}}$. □

For a weighted graph $G = (N, E, w)$, we denote by M the maximum edge-weight, *i.e.*, $M = \max_{e \in E} w(e)$, and by E^M the set of edges of maximum weight in G, *i.e.*, $E^M = \{e \in E \ ; \ w(e) = M\}$. A subset $A \subseteq N$ is *M-maximal* if A is a maximal subset w.r.t. inclusion such that A is connected, and $E(A) \subseteq E^M$. For a vertex $x \in V$, we denote by $N(x)$ its neighborhood (*i.e.*, the set of vertices adjacent to x). The neighborhood of a subset A of N, is defined by $N(A) := \cup_{x \in A} N(x)$. We say that a vertex

$i \in N(A) \setminus A$ is a *neighbor* of the set A, and we denote by $E(A, i)$ the set of edges joining i to A in G.

Lemma 8. *Let $G = (N, E, w)$ be an arbitrary weighted graph and let us consider an M-maximal subset A of N. For all $i \in N(A) \setminus A$, we have $\mathcal{P}_{\min}(A \cup \{i\})_{|A} = \{A\}$.*

Proof. We cannot have $w(e) = M$ for all $e \in E(A, i)$ otherwise A is not M-maximal. Therefore there exists at least one edge $e = \{i, j\} \in E(A, i)$ such that $w(e) < M$. If all edges in $E(A, i)$ have the same weight $w(e)$, then $\mathcal{P}_{\min}(A \cup \{i\}) = \{A, \{i\}\}$. Otherwise $\mathcal{P}_{\min}(A \cup \{i\}) = \{A \cup \{i\}\}$. □

Theorem 9. *Let $G = (N, E, w)$ be an arbitrary weighted graph, and $\mathcal{F}$ the family of connected subsets of N. If for every unanimity game (N, u_S), the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is $\mathcal{F}$-convex, then an M-maximal subset A of N has at most one neighbor in $N \setminus A$.*

Proof. Let us assume A is M-maximal and has two distinct neighbors denoted by 1 and 2 in N. Let us consider $B = A \cup \{2\}$ and $i = 1$. Applying Lemma 8, we have $\mathcal{P}_{\min}(A \cup \{i\})_{|A} = \{A\}$ (resp. $\mathcal{P}_{\min}(B)_{|A} = \{A\}$). Then $\mathcal{P}_{\min}(B)_{|A} = \{A\} \neq \mathcal{P}_{\min}(A)_{|A}$ and it contradicts Theorem 3. □

Let $\gamma = (e_1, e_2 \ldots, e_m)$ be an elementary path of G (*i.e.*, a path with no repeated vertices) with $e_i = \{i, i+1\}$ for $1 \leq i \leq m$. We now establish that we have a property of convexity on the edge-weights along every elementary path in G.

Path Condition. *For every elementary path $\gamma = (1, e_1, 2, e_2, 3, \ldots, m, e_m, m+1)$ in G and for all i, j, k such that $1 \leq i < j < k \leq m$, the edge-weights satisfy:*

$$w_j \leq \max(w_i, w_k).$$

Corollary 10. *Let $G = (N, E, w)$ be a weighted graph, and $\mathcal{F}$ the family of connected subsets of N. If for every unanimity game (N, u_S), the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is $\mathcal{F}$-convex, then the Path Condition is satisfied.*

Proof. It is sufficient to prove that we have:

$$w_j \leq \max(w_1, w_m), \ \forall j, \ 1 \leq j \leq m. \tag{15}$$

Actually we can do the same reasoning with w_i, w_k by considering the restriction of γ between i and $k+1$. Let $G' = (N', E', w)$ be the subgraph of G induced by the vertices of γ. Let us consider an M-maximal subset A in G'. Then A contains the end-vertices of e_1 or e_m otherwise it has two neighbors in $N' \setminus A$, contradicting Theorem 9. Therefore $w_1 = M$ or $w_m = M$ and (15) is satisfied. □

Remark 3. Moreover we have proved that $\max_{e \in E'} w(e) = \max_{e \in \gamma} w(e) = \max(w_1, w_m)$.

We can also obtain necessary conditions for inheritance of convexity in the case of a simple cycle in G (*i.e.*, a cycle with no repeated vertices or edges except for the start and end vertex). For a given cycle C we denote by $E(C)$ the set of edges in E having their end vertices in C.

Cycle Condition. *For every simple cycle of G, $C = (1, e_1, 2, e_2, \ldots, m, e_m, 1)$ with $m \geq 3$, the edge-weights satisfy, after renumbering the edges if necessary:*

$$w_1 \leq w_2 \leq w_3 = \cdots = w_m = M \tag{16}$$

where $M = \max_{e \in E(C)} w(e)$. Moreover $w(e) = w_2$ for all chord incident to 2, and $w(e) = M$ for all chord non incident to 2.

Corollary 11. *Let $G = (N, E, w)$ be a weighted graph, and $\mathcal{F}$ the family of connected subsets of N. If for every unanimity game (N, u_S), the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is $\mathcal{F}$-convex, then the Cycle Condition is satisfied.*

Proof. If $|E(C)| = 3$ then the results are obviously satisfied. Let us assume $|E(C)| \geq 4$. Let us denote by $G' = (N', E', w)$ the subgraph of G induced by the vertices of C. Let A be an M-maximal subset of G'. Theorem 9 implies $|N' \setminus A| \leq 1$, therefore (16) is obviously satisfied, after renumbering the edges if necessary. Moreover as $w(e) = M$ for all $e \in E(A)$, we have $w(e) = M$ for all chord e non incident to 2. Let us now consider a chord $e = \{2, j\}$ incident to 2. Then Proposition 7 applied to the star defined by $\{e_1, e_2, e\}$ implies $w_1 \leq w_2 = w(e)$ or $w(e) \leq w_1 = w_2$. By contradiction let us assume $w(e) < w_1 = w_2$. Using Star condition, we have $w(e') = w_1 = w_2$ for all other chords $e' = \{2, k\}$ incident to 2. If such a chord e' exists we can consider a cycle smaller than C by replacing the vertex 1 or 2 by k. Hence we can assume that e is a unique chord incident to 2. Let us consider $A = V(C) \setminus \{1, 3\}$, $B = V(C) \setminus \{1\}$, and $i = 1$ as represented in Figure 6. Then $\mathcal{P}_{\min}(A \cup \{i\}) = \{A \cup \{i\}\}$, $\mathcal{P}_{\min}(B) = \{B\}$, and $\mathcal{P}_{\min}(A) = \{\{2\}, A \setminus \{2\}\}$. Therefore $\mathcal{P}_{\min}(B)_{|A \cup \{i\}} = \{A\} \neq \mathcal{P}_{\min}(A)_{|A \cup \{i\}}$ and it contradicts Theorem 3. $\square$

Let us consider a weighted graph $G = (N, E, w)$ and a cycle C in G. If $e \in E(C)$ is a chord of C such that $w(e) = \max_{e \in E(C)} w(e)$, we say that e is a *maximum weight chord*.

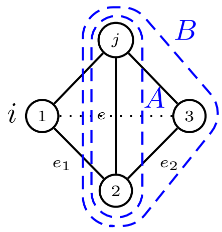

Figure 6: Cycle C with $w(e) < w_1 = w_2$.

Pan Condition. *For all connected subgraphs corresponding to the union of a simple cycle $C = \{e_1, e_2, \ldots, e_m\}$ with $m \geq 3$, and an elementary path P such that there is an edge e in P with $w(e) \leq \min_{1\leq k\leq m} w_k$ and $|V(C) \cap V(P)| = 1$, the edge-weights satisfy:*

$$\text{either} \quad w_1 = w_2 = w_3 = \cdots = w_m = M, \tag{17}$$

$$\text{or} \quad w_1 = w_2 < w_3 = \cdots = w_m = M, \tag{18}$$

where $M = \max_{e\in E(C)} w(e)$. In this last case $V(C) \cap V(P) = \{2\}$ and if moreover $w(e) < w_1$ then $\{1, 3\}$ is a maximum weight chord of C.

Proposition 12. *Let $G = (N, E, w)$ be a weighted graph, and $\mathcal{F}$ the family of connected subsets of N. If for every unanimity game (N, u_S), the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is $\mathcal{F}$-convex, then the Pan Condition is satisfied.*

Proof. Let us consider $C = \{1, e_1, 2, e_2, 3, \ldots, m, e_m, 1\}$, and $P = \{j, e_{m+1}, m + 1, e_{m+2}, m + 2, \ldots, e_{m+r}, m + r\}$ with $j \in \{1, \ldots, m\}$, as represented in Figure 7. We can assume w.l.o.g. that $e = e_{m+r}$ (restricting P if necessary) and that $w_{m+j} > w_{m+r} = w(e)$ for all $1 \leq j \leq r - 1$. Applying Corollary 11

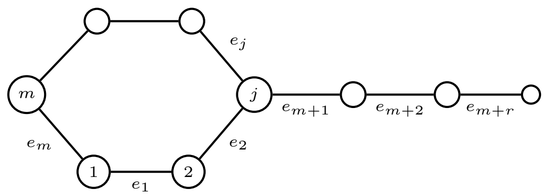

Figure 7: Pan formed by the union of C and P.

to the cycle C, we have after renumbering the edges if necessary:

$$w_1 \leq w_2 \leq w_3 = \cdots = w_m = M. \tag{19}$$

a) Let us first assume $3 \leq j \leq m$. Applying Corollary 10 to the path $\{2, e_1, 1, e_m, m, \ldots, j + 1, e_j, j, e_{m+1}, m + 1, \ldots, m + r - 1, e_{m+r}, m + r\}$, we have $w_j \leq \max(w_1, w(e)) = w_1$. Then (19) implies (17).

b) Let us now assume $j \in \{1, 2\}$. If $r = 1$ then $w_{m+1} = w(e) \le w_1$. Otherwise, applying Corollary 10 to the path $\{e_1, e_{m+1} \ldots, e_{m+r}\}$, we have $w_{m+1} \le \max(w_1, w(e)) = w_1$.
If $j = 1$, Proposition 7 applied to the star defined by $\{e_1, e_m, e_{m+1}\}$, implies $w_{m+1} \le w_1 = w_m$. Hence (19) still implies (17).
If $j = 2$, Proposition 7 applied to the star defined by $\{e_1, e_2, e_{m+1}\}$, implies $w_{m+1} \le w_1 = w_2$. If $w_1 = w_2 = M$ then (17) is satisfied. Otherwise we have $w_1 = w_2 < M$ and therefore (18) is satisfied. In this last case let us assume by contradiction that $\{1, 3\} \notin E(C)$. Corollary 11 implies $w(e) = M$ (resp. $w(e) = w_2$) for any chord e of C non incident (resp. incident) to 2. Therefore we can assume w.l.o.g. that C has no maximum weight chord (otherwise we can replace C by a smaller cycle which still contains the vertices $1, 2, 3$). Let us consider $i \in V(C) \setminus \{1, 2, 3\}$, $A = V(C) \setminus \{i\}$ and $B = A \cup V(P)$ as represented in Figure 8. Then $\mathcal{P}_{\min}(A) =$

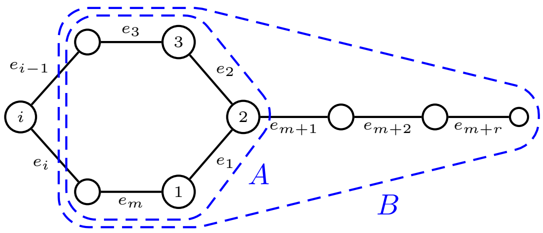

Figure 8: $w_{m+r} < w_1 = w_2 < w_3 = \cdots = w_m = M$.

$\{\{2\}, \{3, 4, \ldots, i-1\}, \{i+1, \ldots, m, 1\}\}$, $\mathcal{P}_{\min}(A \cup \{i\}) = \{A \cup \{i\} \setminus \{2\}, \{2\}\}$, and $\mathcal{P}_{\min}(B) = \{B \setminus \{m+r\}, \{m+r\}\}$ or $\mathcal{P}_{\min}(B) = \{B\}$ (this last case can occur if there exists an edge $e^{'}$ in G_B with $w(e^{'}) < w(e)$ or with $m+r$ as an end-vertex and $w(e^{'}) > w(e)$). Therefore $A^{'} := A \cup \{i\} \setminus \{2\} \in \mathcal{P}_{\min}(A \cup \{i\})$, but $\mathcal{P}_{\min}(B)_{|A^{'}} = \{A \setminus \{2\}\} \neq \mathcal{P}_{\min}(A)_{|A^{'}}$ and it contradicts Theorem 3. $\square$

Let $G = (N, E, w)$ be a graph. We say that two cycles are adjacent if they have at least one common edge. Let us consider a cycle $C = (1, e_1, 2, e_2, \ldots, m, e_m, 1)$ such that $w_1 \le w_2 \le w_3 = \ldots = w_m$. If $w_1 < w_3$ (resp. $w_2 < w_3$), we say that e_1 (resp. e_2) is a non-maximum weight edge of C.

Lemma 13. *Let $G = (N, E, w)$ be a weighted graph satisfying the Star and Cycle conditions. Then for all pairs $(C, C^{'})$ of adjacent simple cycles in G, we have:*

$$M = \max_{e \in E(C)} w(e) = \max_{e \in E(C^{'})} w(e) = M^{'}. \tag{20}$$

Proof. Let us consider two adjacent cycles C and $C^{'}$ with $M < M^{'}$. There is at least one edge e_1 common to C and $C^{'}$. Then we have $w_1 \le M < M^{'}$ and therefore e_1 is a non-maximum weight edge in $C^{'}$. The cycle condition implies that there are at most two non-maximum weight edges in $C^{'}$ (cf. Corollary 11). Therefore there exists an edge $e_2^{'}$ in $C^{'}$ adjacent to e_1 with $w_2^{'} = M^{'}$. As $M^{'} > M$, $e_2^{'}$ is not an edge of C. Let e_2 be the edge of C adjacent to e_1 and $e_2^{'}$. Then we have $w_2 \leq M < M^{'}$ but it contradicts the star condition applied to $\{e_1, e_2, e_2^{'}\}$ (two edge-weights are strictly smaller than $w_2^{'}$). □

Adjacent Cycles Condition. *For all pairs $(C, C^{'})$ of adjacent simple cycles in G such that:*

1. *$V(C) \setminus V(C^{'}) \neq \emptyset$ and $V(C^{'}) \setminus V(C) \neq \emptyset$,*
2. *C has at most one non-maximum weight chord,*
3. *C and $C^{'}$ have no maximum weight chord,*
4. *C and $C^{'}$ have no common chord,*

then C and $C^{'}$ cannot have two common non-maximum weight edges. Moreover C and $C^{'}$ have a unique common non-maximum weight edge e_1 if and only if there are non-maximum weight edges $e_2 \in E(C) \setminus E(C^{'})$ and $e_2^{'} \in E(C^{'}) \setminus E(C)$ such that $e_1, e_2, e_2^{'}$ are adjacent and:

- *$w_1 = w_2 = w_2^{'}$ if $|E(C)| \geq 4$ and $|E(C^{'})| \geq 4$.*
- *$w_1 = w_2 \geq w_2^{'}$ or $w_1 = w_2^{'} \geq w_2$ if $|E(C)| = 3$ or $|E(C^{'})| = 3$.*

Proposition 14. *Let $G = (N, E, w)$ be a weighted graph, and $\mathcal{F}$ the family of connected subsets of N. Let us assume that for every unanimity game (N, u_S), the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is $\mathcal{F}$-convex. Then the Adjacent Cycles Condition is satisfied.*

Proof of Proposition 14. Let us consider two adjacent cycles $C = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$ and $C^{'} = \{1^{'}, e_1^{'}, 2^{'}, e_2^{'}, \ldots, p^{'}, e_p^{'}, 1^{'}\}$. By Lemma 13 we have $M = \max_{e \in E(C)} w(e) = \max_{e \in E(C^{'})} w(e) = M^{'}$. Corollary 11 implies that there are at most two non-maximum weight edges in $E(C) \cap E(C^{'})$.

A) Let us first assume that there are exactly two common non-maximum weight edges e_1, e_2. Therefore we can assume $e_1 = e_1^{'}$, $e_2 = e_2^{'}$, $j = j^{'}$ for $1 \leq j \leq 3$. Of course C and $C^{'}$ may have other common edges or vertices. Hence we have:

$$w_1 \leq w_2 < M = \max_{e \in E(C)} w(e) = \max_{e \in E(C^{'})} w(e) = M^{'}. \tag{21}$$

Corollary 11 also implies that all chords of C and $C^{'}$ are incident to 2 and have weight w_2. By assumption, C has at most one non-maximum weight chord in $E(C) \setminus E(C^{'})$. If C has one non-maximum weight chord

e in $E(C) \setminus E(C^{'})$ then $e = \{2, i\}$ with $i \in V(C) \setminus V(C^{'})$. Let us define $A = V(C) \setminus \{i\}$, $B = A \cup V(C^{'})$ as represented in Figure 9 with $m = p = 6$. If C has no chord, then we choose $i \in V(C) \setminus V(C^{'})$ arbitrarily. We now

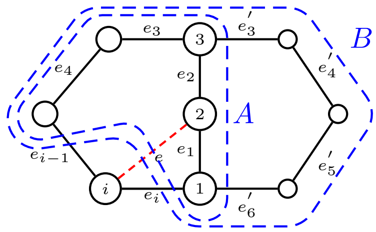

Figure 9: C and $C^{'}$ with two common non-maximum weight edges.

consider two cases.
Case 1 If $w_1 < w_2$ then $\mathcal{P}_{\min}(A) = \{\{2, 3, \ldots, i-1\}, \{i+1, \ldots, m, 1\}\}$, $\mathcal{P}_{\min}(A \cup \{i\}) = \{A \cup \{i\}\}$, and $\mathcal{P}_{\min}(B) = \{B\}$. Therefore $A^{'} := A \cup \{i\} \in \mathcal{P}_{\min}(A \cup \{i\})$, but $\mathcal{P}_{\min}(B)_{|A'} = \{A\} \neq \mathcal{P}_{\min}(A)_{|A'}$ and it contradicts Theorem 3.
Case 2 Now if $w_1 = w_2$, we have $\mathcal{P}_{\min}(A) = \{\{2\}, \{3, 4, \ldots, i-1\}, \{i+1, \ldots, m, 1\}\}$, $\mathcal{P}_{\min}(A \cup \{i\}) = \{A \cup \{i\} \setminus \{2\}, \{2\}\}$, and $\mathcal{P}_{\min}(B) = \{B \setminus \{2\}, \{2\}\}$ (as $V(C^{'}) \setminus V(C) \neq \emptyset$). Taking $A^{'} = A \cup \{i\} \setminus \{2\}$, we have $\mathcal{P}_{\min}(B)_{|A'} = \{A \setminus \{2\}\} \neq \mathcal{P}_{\min}(A)_{|A'}$, and it still contradicts Theorem 3.

B) Let us now assume that there is exactly one common non-maximum weight edge e_1. Therefore we can assume $e_1 = e_1^{'}$, $j = j^{'}$ for $1 \leq j \leq 2$, and we have:

$$w_1 < M = \max_{e \in E(C)} w(e) = \max_{e \in E(C^{'})} w(e) = M^{'}. \tag{22}$$

If the edges in C (resp. C') adjacent to e_1 are maximum weight edges then Corollary 11 applied to C (resp. $C^{'}$) implies $w(e) = M$ for all $e \in E(C) \setminus \{e_1\}$ (resp. $e \in E(C^{'}) \setminus \{e_1\}$). Therefore C and $C^{'}$ have no chord. We choose arbitrarily a vertex $i \in V(C) \setminus V(C^{'})$. Then taking $A = V(C) \setminus \{i\}$, and $B = A \cup V(C^{'})$, we get the same contradiction as in Case 1. If there is a non-maximum weight edge e in C or $C^{'}$ adjacent to e_1 then we can assume w.l.o.g. $e = e_2 = \{2, 3\} \in C$. Let us consider $e_2^{'} = \{2, 3^{'}\} \in C^{'}$. We necessarily have $3 \neq 3^{'}$ otherwise $e_2^{'} = e_2$ and e_1, e_2 would be two common non-maximum weight edges, a contradiction. If $e_2^{'}$ is a maximum weight edge then it contradicts Proposition 7 applied to the star defined by $\{e_1, e_2, e_2^{'}\}$. Therefore $e_2^{'}$ is a non-maximum weight edge. Proposition 7 applied to the star defined by $\{e_1, e_2, e_2^{'}\}$ implies $w_1 \leq w_2 = w_2^{'}$ or $w_2^{'} \leq w_1 = w_2$ or $w_2 \leq w_1 = w_2^{'}$. Let us first assume $w_1 < w_2 = w_2^{'}$. Then we can establish the same contradiction as in Case 1. Hence Proposition 14 is satisfied if

$E(C) = 3$ or $E(C^{'}) = 3$. Let us now assume $w_2^{'} < w_1 = w_2$. If $|E(C)| \geq 4$ then, as C has no chord, it contradicts Proposition 12 applied to the pan defined by C and $e_2^{'}$ and represented in Figure 10. By symmetry the case

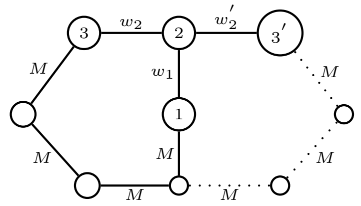

Figure 10: $w_2^{'} < w_1 = w_2 < M$ contradicts the pan condition.

$w_2 < w_1 = w_2^{'}$ is also impossible if $|E(C^{'})| \geq 4$. Finally if e_1, e_2 and $e_2^{'}$ satisfies the conditions of the proposition then e_1 is the unique common non-maximum weight edge as each cycle cannot have more than two non-maximum weight edges. $\square$

Remark 4. We consider simple cycles in Proposition 14 to avoid situations similar to the one represented in Figure 11. In this case we cannot establish

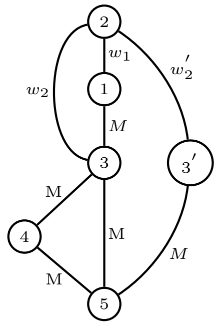

Figure 11: $C = \{1, e_1, 2, e_2, 3, e_3, 4, e_4, 5, e_5, 3, e_6, 1\}$, $C^{'} = \{1, e_1, 2, e_2^{'}, 3^{'}, e_3^{'}, 5, e_5, 3, e_6, 1\}$ with $w_1 < w_2 = w_2^{'} < M$ but C is not a simple cycle.

the same contradiction as in the proof of Proposition 14 even though $V(C) \setminus V(C^{'}) \neq \emptyset$ and $V(C^{'}) \setminus V(C) \neq \emptyset$.

5.2 Sufficient conditions

Let $\mathcal{F}$ be the family of connected subsets of N. We will now prove that the preceding necessary conditions are also sufficient for superadditive games. We first need some useful lemmas. The first one was already proved in [10].

Lemma 15. *Let us consider subsets $A, B \subseteq N$ and a partition $\{B_1, B_2, \ldots, B_p\}$ of B. If A, B_i, and $A \cap B_i \in \mathcal{F}$, for all $i \in \{1, \ldots, p\}$, then for every $\mathcal{F}$-convex game (N, v) we have:*

$$v(A \cup B) + \sum_{i=1}^{p} v(A \cap B_i) \geq v(A) + \sum_{i=1}^{p} v(B_i). \tag{23}$$

Lemma 16. *Let us consider a correspondence $\mathcal{P}$ on N and subsets $A \subseteq B \subseteq N$ such that $\mathcal{P}(A) = \mathcal{P}(B)_{|A}$. If $A \in \mathcal{F}$ and if all elements of $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are in $\mathcal{F}$, then for every $\mathcal{F}$-convex game (N, v) we have:*

$$v(B) - \overline{v}(B) \geq v(A) - \overline{v}(A). \tag{24}$$

Proof. If $\mathcal{P}(B) = \{B_1, B_2, \ldots, B_p\}$ then $\mathcal{P}(A) = \{A \cap B_1, A \cap B_2, \ldots, A \cap B_p\}$, and Lemma 15 implies (24). $\square$

We say that an edge $e \in E$ is connected to a subset $A \subseteq N$, if there is a path in G joining e to A. The following lemma gives simple conditions ensuring $\mathcal{P}_{\min}(A)$ is induced by $\mathcal{P}_{\min}(B)$ for $A \subseteq B$.

Lemma 17. *Let $G = (N, E, w)$ be a weighted graph, and $\mathcal{F}$ the family of connected subsets of N. Let us assume that the edge-weight function w satisfies the Pan condition. Let us consider $A, B \in \mathcal{F}$ such that $A \subseteq B \subseteq N$ and $\sigma(A) = \sigma(B)$. Let us assume that either the subgraph $G_B = (B, E(B))$ is cycle-free or there exists an edge $e \in E$ connected to B with $w(e) < \sigma(B)$. Then $\mathcal{P}_{\min}(A) = \mathcal{P}_{\min}(B)_{|A}$ and for every $\mathcal{F}$-convex game (N, v) we have:*

$$v(B) - \overline{v}(B) \geq v(A) - \overline{v}(A). \tag{25}$$

Proof. Let B_k be a component of $\mathcal{P}_{\min}(B)$ such that $A \cap B_k \neq \emptyset$. Let α_0 be a fixed vertex of $A \cap B_k$ and A_k be the component of $\mathcal{P}_{\min}(A)$ which contains α_0. We will prove $A \cap B_k = A_k$. As $\sigma(A) = \sigma(B)$, $\Sigma(A) = E(A) \cap \Sigma(B)$. Let α_1 be another vertex of A_k and γ be a path in A_k from α_0 to α_1. Each edge e of γ is in $E(A) \setminus \Sigma(A)$ and therefore also in $E(B) \setminus \Sigma(B)$. Hence γ is also a path from α_0 to α_1 in B and therefore $\alpha_1 \in B_k$. That is:

$$A_k \subseteq A \cap B_k. \tag{26}$$

Let us assume there is a vertex α_1 in $A \cap B_k \setminus A_k$. As A (resp. B_k) is connected, there exists a path γ (resp. γ') from α_0 to α_1 in A (resp. B_k). By definition of B_k, we have $w(e) > \sigma(A) = \sigma(B)$ for all $e \in \gamma'$. As $\alpha_1 \notin A_k$, there exists at least one edge $e' \in \gamma$ such that $w(e') = \sigma(A) = \sigma(B)$. Therefore $\gamma \neq \gamma'$ and :

$$w(e) > w(e'), \ \forall e \in \gamma'. \tag{27}$$

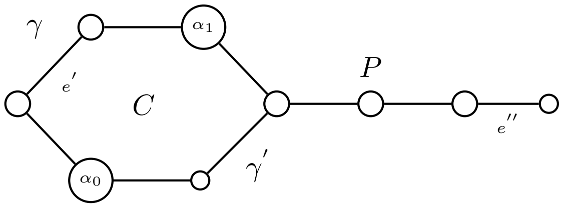

Figure 12: $w(e) > w(e^{'})$, $\forall e \in \gamma^{'}$, and $w(e^{''}) < \min_{e\in E(C)} w(e)$.

Hence γ and $\gamma^{'}$ form a cycle C in G_B. If G_B is cycle-free we get a contradiction. Otherwise we can select α_0, α_1 and $\gamma, \gamma^{'}$ such that C is a simple cycle without maximum weight chord (γ and $\gamma^{'}$ are paths with minimum number of edges). By assumption, there exists an edge $e^{''} \in E$ connected to B such that $w(e^{''}) < \sigma(B)$. Therefore there is a path P linking $e^{''}$ to C as represented in Figure 12 and we have $w(e^{''}) < \min_{e\in E(C)} w(e)$. If $|V(C)| \geq 4$ or if $|V(C)| = 3$ and $\gamma^{'}$ contains two edges then (27) contradicts the pan condition. Otherwise $\gamma^{'}$ is reduced to the edge $e = \{\alpha_0, \alpha_1\}$ and therefore (27) implies $\alpha_1 \in A_k$, a contradiction. Finally Lemma 16 implies (25). □

Lemma 18. *Let $G = (N, E, w)$ be a weighted graph, and $\mathcal{F}$ the family of connected subsets of N. Let us assume that the edge-weight function w satisfies the Path and Star conditions. For a given $i \in N$, if $A, B \in \mathcal{F}$, $A \subseteq B \subseteq N \setminus \{i\}$, and $E(A, i) \neq \emptyset$ then either $\sigma(A, i) \geq \sigma(A) \geq \sigma(B)$ or $\sigma(A) = \sigma(B) > \sigma(A, i)$ where $\sigma(A, i) = \min_{e\in E(A,i)} w(e)$.*

Proof. As $A \subseteq B$, we have $\sigma(A) \geq \sigma(B)$. Let us assume:

$$\sigma(A) > \sigma(A, i). \tag{28}$$

Let $e = \{i, j\}$ be an edge in $E(A, i)$ such that $w(e) = \sigma(A, i)$. As A (resp. B) is connected, there exists an elementary path $\gamma = (e_1, e_2, \ldots, e_m)$ in G_A (resp. $\gamma^{'} = (e^{'}_1, e^{'}_2, \ldots, e^{'}_r)$ in G_B) with $w_1 = \sigma(A)$ (resp. $w^{'}_1 = \sigma(B)$) and such that j is an end-vertex of e_m (resp. $e^{'}_r$), as represented in Figure 13. If γ

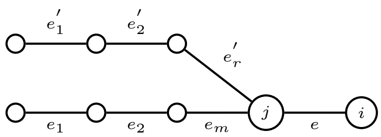

Figure 13: $w(e) = \sigma(A, i)$, $w_1 = \sigma(A)$, $w^{'}_1 = \sigma(B)$.

is reduced to e_1 then (28) implies $w_1 > w(e)$. Otherwise the path condition applied to $\gamma \cup \{e\}$ and (28) imply $w_m \leq \max(w_1, w(e)) = w_1 = \sigma(A)$. As $e_m \in E(A)$, $w_m = \sigma(A)$ and using again (28), we obtain:

$$w_m > w(e). \tag{29}$$

If $e_r^{'} = e_m$, we have $w_r^{'} = w_m = \sigma(A)$. Otherwise, the star condition applied to $\{e_r^{'}, e_m, e\}$ and (29) imply again $w_r^{'} = w_m = \sigma(A)$. If $\gamma^{'}$ is reduced to $e_1^{'}$ then $w_1^{'} = \sigma(A)$ and therefore $\sigma(B) = \sigma(A)$. Otherwise the path condition applied to $\gamma^{'} \cup \{e\}$ implies $w_r^{'} \leq \max(w_1^{'}, w(e))$ and therefore:

$$\sigma(A) \leq \max(\sigma(B), \sigma(A,i)). \tag{30}$$

Then (28) and (30) imply $\sigma(A) \leq \sigma(B)$. Therefore $\sigma(A) = \sigma(B)$. □

Lemma 19. *Let $G = (N, E, w)$ be a weighted graph, and $\mathcal{F}$ the family of connected subsets of N. Let us assume that the edge-weight function w satisfies the Pan condition. For $i \in N$, and $A, B \in \mathcal{F}$, $A \subseteq B \subseteq N \setminus \{i\}$, let $A_1, A_2, \ldots, A_k$ be the components of $\mathcal{P}_{\min}(A)$ and let us assume $\sigma(A) > \sigma(B)$. Then for every edge $e = \{i, j\}$ in $E(A, i)$ such that $w(e) > \sigma(A)$, j belongs to the same component A_1, after renumbering if necessary.*

Proof. Let us assume w.l.o.g. that there exists an edge $e_1 = \{i, j_1\}$ (resp. $e_2 = \{i, j_2\}$) with $j_1 \in A_1$ (resp. $j_2 \in A_2$) such that $w_1 > \sigma(A)$ (resp. $w_2 > \sigma(A)$). As A is connected there exists a path γ in A from j_1 to j_2. We obtain a cycle $C = \{i, j_1\} \cup \gamma \cup \{j_2, i\}$. There exists at least one edge of weight $\sigma(A)$ in γ and $w(e) \geq \sigma(A)$ for all $e \in \gamma$. We can select A_1, A_2 and j_1, j_2 such that there is no maximum weight chord in C. As B is connected and as $\sigma(A) > \sigma(B)$, there is an edge e in $E(B)$ of weight $\sigma(B)$ linked by a path to C as represented in Figure 14, but it contradicts the Pan condition. □

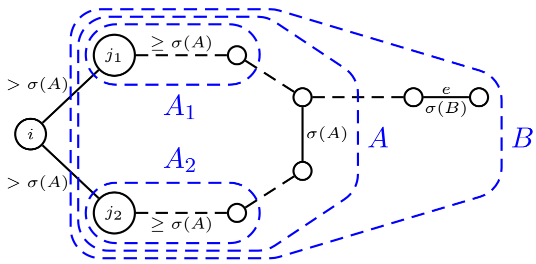

Figure 14: $|V(C)| \geq 3$ and $\sigma(A) > \sigma(B)$.

Remark 5. Lemma 19 is also valid with $\sigma(A) = \sigma(B)$ if there exists an edge $e \in E(B, i) \setminus E(A, i)$ such that $w(e) < \sigma(A)$.

We can now establish that the Path, Star, Cycle, Pan and Adjacent cycles conditions defined in Section 5.1 are sufficient for superadditive games.

Theorem 20. *Let $G = (N, E, w)$ be a weighted graph. For every superadditive and $\mathcal{F}$-convex game (N, v), the $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$ is $\mathcal{F}$-convex if and only if the Path, Star, Cycle, Pan, and Adjacent cycles conditions are satisfied.*

Proof. We have already proved that the conditions are necessary. We now prove they are sufficient. Let (N, v) be a given $\mathcal{F}$-convex game. According to Theorem 2, we have to prove that, for all $i \in N$, for all $A \subseteq B \subseteq N \setminus \{i\}$ and $A, B, A \cup \{i\} \in \mathcal{F}$, we have:

$$(31) \qquad \overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A).$$

As $A \cup \{i\}$ is connected, there exists an edge $e = \{i, j\}$ with $j \in A$. As $E(A, i) \subseteq E(B, i)$, we have $\sigma(A, i) \geq \sigma(B, i)$. Using Lemma 18, we have several cases to consider.

Case 1 $\sigma(A) = \sigma(B) > \sigma(A, i)$.

Case 1.1 Let us first assume there exists an edge $e \in E(A, i)$ such that $w(e) > \sigma(A, i)$. Then $\mathcal{P}_{\min}(A \cup \{i\}) = \{A \cup \{i\}\}$ (resp. $\mathcal{P}_{\min}(B \cup \{i\}) = \{B \cup \{i\}\}$) as edges in $E(A)$ (resp. $E(B)$) and e are not deleted. Hence (31) is equivalent to $v(B \cup \{i\}) - \overline{v}(B) \geq v(A \cup \{i\}) - \overline{v}(A)$. This last inequality is satisfied as Lemma 17 implies $v(B) - \overline{v}(B) \geq v(A) - \overline{v}(A)$ and the $\mathcal{F}$-convexity of v implies $v(B \cup \{i\}) - v(B) \geq v(A \cup \{i\}) - v(A)$.

Case 1.2 Let us now assume that for all $e \in E(A, i)$, $w(e) = \sigma(A, i)$. We consider several subcases.

Case 1.2.1 Let us assume that there exists an edge $e \in E(B, i)$ such that $w(e) > \sigma(B, i)$. Then $\mathcal{P}_{\min}(A \cup \{i\}) = \{A, \{i\}\}$ (all edges in $E(A, i)$ are deleted) and $\mathcal{P}_{\min}(B \cup \{i\}) = \{B \cup \{i\}\}$ (edges in $E(B)$ and e are not deleted). Then (31) is equivalent to $v(B \cup \{i\}) - \overline{v}(B) \geq v(A) + v(i) - \overline{v}(A)$. This last inequality is satisfied as Lemma 17 implies $v(B) - \overline{v}(B) \geq v(A) - \overline{v}(A)$ and the superadditivity of v implies $v(B \cup \{i\}) - v(B) \geq v(i)$.

Case 1.2.2 Let us now assume that for all $e \in E(B, i)$, $w(e) = \sigma(B, i)$. Then $\mathcal{P}_{\min}(A \cup \{i\}) = \{A, \{i\}\}$ and $\mathcal{P}_{\min}(B \cup \{i\}) = \{B, \{i\}\}$. Therefore (31) is equivalent to $v(B) - \overline{v}(B) \geq v(A) - \overline{v}(A)$. Lemma 17 implies that this inequality is satisfied.

Case 2 $\sigma(A, i) \geq \sigma(A) = \sigma(B)$.

If $w(e) = \sigma(A)$ for all $e \in E(A, i)$ then $\mathcal{P}_{\min}(A \cup \{i\}) = \{\mathcal{P}_{\min}(A), \{i\}\}$, and (31) is equivalent to $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A) + v(i) - \overline{v}(A) = v(i)$. Corollary 4 implies the superadditivity of $\overline{v}$ and therefore the last inequality is satisfied. Otherwise, let $A_1, A_2, \ldots, A_k$ (resp. $B_1, B_2, \ldots, B_l$) be the components of $\mathcal{P}_{\min}(A)$ (resp. $\mathcal{P}_{\min}(B)$) connected to i by edges in $E(A, i)$ (resp. $E(B, i)$) with weights strictly greater than $\sigma(A)$. Then (31) is equivalent to

$$(32) \quad v(B_1 \cup \cdots \cup B_l \cup \{i\}) - \sum_{j=1}^{l} v(B_j) \geq v(A_1 \cup \cdots \cup A_k \cup \{i\}) - \sum_{j=1}^{k} v(A_j).$$

As $A \subseteq B$ and as $\sigma(A) = \sigma(B)$, each A_j, $1 \leq j \leq k$ is a subset of some B_p, $1 \leq p \leq l$. Let us prove that if $j_1 \neq j_2$, then $A_{j_1} \subseteq B_{j_1}$ and $A_{j_2} \subseteq B_{j_2}$ with $B_{j_1} \neq B_{j_2}$. By contradiction, let us assume that A_1 and A_2 are subsets of B_1, after renumbering if necessary. There exist edges $\{i, j_1\}$ and $\{i, j_2\}$ with $j_1 \in A_1$, $j_2 \in A_2$, and weights strictly greater than $\sigma(A)$. As A is connected there exists a path γ in A from j_1 to j_2. We obtain a cycle $C_1 = \{i, j_1\} \cup \gamma \cup \{j_2, i\}$. Let us select j_1, j_2 and γ such that C_1 is of minimum length. Hence C_1 has no chord except possibly between i and vertices of γ. As the components A_1 and A_2 in $\mathcal{P}_{\min}(A)$ are obtained by deleting edges of weight $\sigma(A)$ in $E(A)$, there exists at least one edge in γ of weight $\sigma(A)$. The cycle condition applied to C_1 implies there are two adjacent edges e_1, e_2 with $\sigma(A) = w_1 < w_2 \leq M$ or $\sigma(A) = w_1 = w_2 < M$ and all other edges in γ have weight $M = \max_{e \in E(C_1)} w(e)$ (Note that if $|E(C_1)| = 3$ then we necessarily have $\sigma(A) = w_1 < w_2 \leq M$ with e_2 incident to i). Hence we can assume $A_1 \cap C_1 = \{2, 3, \ldots, j_1 = i - 1\}$ if $\sigma(A) = w_1 < w_2 \leq M$, or $A_1 \cap C_1 = \{3, \ldots, j_1 = i - 1\}$ if $\sigma(A) = w_1 = w_2 < M$, and $A_2 \cap C_1 = \{j_2 = i + 1, i + 2, \ldots, m, 1\}$ as represented in Figure 15. If there is a chord $\{i, j\}$ with $j \in A_1$ or $j \in A_2$, then it contradicts the minimality of C_1. Therefore C_1 has at most one chord $\{i, 2\}$ adjacent to e_1 and e_2 (if $w_1 = w_2 < M$).

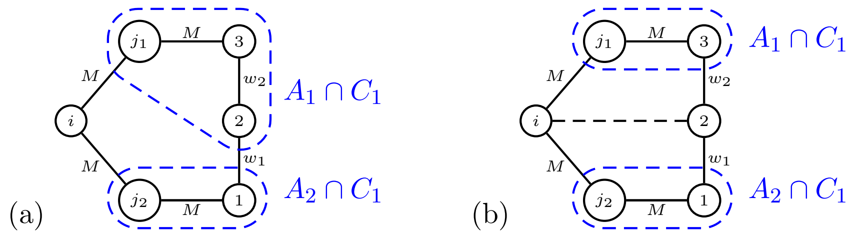

Figure 15: (a) : $w_1 < w_2 \leq M$, (b) : $w_1 = w_2 < M$.

As A_1 and A_2 are both subsets of the same component $B_1 \in \mathcal{P}_{\min}(B)$, there exists a minimum path γ' in B_1 linking $1 \in A_2$ to $k \in A_1$ with $k = 2$ if $w_1 < w_2$ and $k = 3$ if $w_1 = w_2$. By definition of B_1 each edge e' in γ' has a weight $w(e') > \sigma(B) = \sigma(A)$. We get a cycle $C_2 = \{1, e_1, 2, \ldots, k\} \cup \gamma'$. Let us remark that if $w_1 = w_2$ then 2 cannot be a vertex of γ', otherwise there is an edge e' in γ' incident to 2 with $w(e') > w_1 = w_2$, and it contradicts the star condition applied to $\{e_1, e_2, e'\}$ (cf. Figure 16).

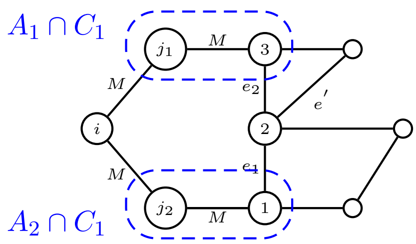

Figure 16: $w_1 = w_2 < w(e')$ contradicts the star condition.

Let us first assume $k = 3$ (*i.e.*, $w_1 = w_2$). Then we necessarily have $|E(C_1)| \geq 4$ and $|E(C_2)| \geq 4$ (otherwise there is an edge from 1 to 3 with weight strictly greater than $\sigma(A)$, and A_1, A_2 are not disjoint components of $\mathcal{P}_{\min}(A)$). The cycle condition applied to C_2 and Lemma 13 imply that all edges $e^{'} \in \gamma^{'}$ have weight $w(e^{'}) = M$. Any chord in C_2 non incident to 2 would contradict the minimality of $\gamma^{'}$. Moreover the cycle condition applied to C_1 (resp. C_2) implies that any chord e in C_1 (resp. C_2) incident to 2 satisfies $w(e) = w_2$. Therefore C_1 and C_2 have no maximum chord. But C_1 and C_2 have two common non-maximum weight edges e_1, e_2, contradicting the adjacent cycles condition.
Let us now assume $k = 2$ (*i.e.*, $w_1 < w_2$). Then C_1 has no chord. C_2 has no chord, otherwise it contradicts the minimality of $\gamma^{'}$. Let $e^{'}$ (resp. $e^{''}$) be the edge of $\gamma^{'}$ incident to 1 (resp. 2) as represented in Figure 17. If

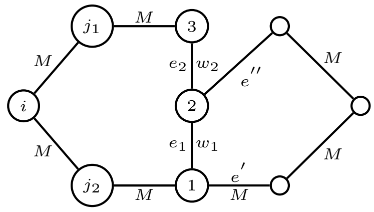

Figure 17: $w_1 < w_2 \leq M$ and $e^{'}$ (resp. $e^{''}$) may coincide with e_m (resp. e_2).

$e^{'} \in C_1$ (resp. $e^{''} \in C_1$) then $w(e^{'}) = M$ (resp. $w(e^{''}) = w_2$). Otherwise, as $w_1 < w_2 \leq M$, the star condition applied to $\{e_1, e_m, e^{'}\}$ (resp. $\{e_1, e_2, e^{''}\}$) implies $w(e^{'}) = M$ (resp. $w(e^{''}) = w_2$). Then the cycle condition applied to C_2 and Lemma 13 imply $w_1 < w(e^{''}) \leq M$ and all other edges in C_2 have weight M. If $w(e^{''}) < M$ and $e^{''} = e_2$ then e_1 and e_2 are two non-maximum common edges, contradicting the adjacent cycles condition. Otherwise e_1 is a unique non-maximum edge common to C_1 and C_2, but all edges adjacent to e_1 have a weight strictly greater than w_1, contradicting the adjacent cycles condition.
Therefore we can assume that $A_j \subseteq B_j \cap A$, $1 \leq j \leq k \leq l$. As $\mathcal{P}_{\min}(A)$ is partition of A, we have $A_j = B_j \cap A$, $1 \leq j \leq k \leq l$. Applying Lemma 16 to $A_1 \cup A_2 \cup \ldots \cup A_k$, we get $v(B_1 \cup \ldots \cup B_k) - \sum_{j=1}^{k} v(B_j) \geq v(A_1 \cup \ldots \cup A_k) - \sum_{j=1}^{k} v(A_j)$. The superadditivity of v implies: $v(B_1 \cup \ldots \cup B_l \cup \{i\}) - \sum_{j=1}^{l} v(B_j) \geq v(B_1 \cup \ldots \cup B_k \cup \{i\}) - \sum_{j=1}^{k} v(B_j)$. And the $\mathcal{F}$-convexity of v implies $v(B_1 \cup \ldots \cup B_k \cup \{i\}) - v(B_1 \cup \ldots \cup B_k) \geq v(A_1 \cup \ldots \cup A_k \cup \{i\}) - v(A_1 \cup \ldots \cup A_k)$. These last inequalities imply (32).

Case 3 $\sigma(A, i) \geq \sigma(A) > \sigma(B)$. If $w(e) = \sigma(A)$ for all $e \in E(A, i)$ then $\mathcal{P}_{\min}(A \cup \{i\}) = \{\mathcal{P}_{\min}(A), \{i\}\}$, and (31) is equivalent to $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A) + v(i) - \overline{v}(A) = v(i)$. Corollary 4 implies the superadditivity of $\overline{v}$ and therefore the last inequality is satisfied. Otherwise we use Lemma 19. It implies that a unique component A_1 of $\mathcal{P}_{\min}(A)$ is connected to i by edges in $E(A,i)$ of weight $> \sigma(A)$. If $\sigma(B,i) \geq \sigma(B)$, then let $B_1, B_2, \ldots, B_l$ be the components of $\mathcal{P}_{\min}(B)$ connected to i by edges in $E(B,i)$ of weight strictly greater than $\sigma(B)$. Then (31) is equivalent to :

$$v(B_1 \cup \ldots \cup B_l \cup \{i\}) - \sum_{j=1}^{l} v(B_j) \geq v(A_1 \cup \{i\}) - v(A_1). \tag{33}$$

As $\sigma(A) > \sigma(B)$, we can assume $A_1 \subseteq B_1$, after renumbering if necessary. Then the $\mathcal{F}$-convexity of v implies $v(B_1 \cup \{i\}) - v(B_1) \geq v(A_1 \cup \{i\}) - v(A_1)$. The superadditivity of v implies $v(B_1 \cup \ldots \cup B_l \cup \{i\}) - \sum_{j=1}^{l} v(B_j) \geq v(B_1 \cup \{i\}) - v(B_1)$. These last inequalities imply (33). Now if $\sigma(B,i) < \sigma(B)$, then let $B_1, B_2, \ldots, B_l$ be all the components of $\mathcal{P}_{\min}(B)$. Then (31) is still equivalent to (33). □

Remark 6. As the example given in Section 3 satisfies the Path condition, it satisfies inheritance of $\mathcal{F}$-convexity.

Corollary 21. *Let $G = (N, E, w)$ be a weighted graph. The following properties are equivalent:*

1) *For every unanimity game (N, u_S), the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is $\mathcal{F}$-convex.*

2) *For every superadditive and $\mathcal{F}$-convex game (N, v), the $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$ is $\mathcal{F}$-convex.*

3) *For all $A, B \in \mathcal{F}$ such that $A \cap B \in \mathcal{F}$, $\mathcal{P}_{\min}(A \cap B) = \{A_j \cap B_k; A_j \in \mathcal{P}_{\min}(A), B_k \in \mathcal{P}_{\min}(B)$ s.t. $A_j \cap B_k \neq \emptyset\}$.*

4) *For all $i \in N$, for all $A \subset B \subseteq N \setminus \{i\}$ with $A, B, A \cup \{i\} \in \mathcal{F}$, and for all $A' \in \mathcal{P}_{\min}(A \cup \{i\})$, $\mathcal{P}_{\min}(B)_{|A'} = \mathcal{P}_{\min}(A)_{|A'}$.*

Proof. Let us assume 1) is satisfied. Then Proposition 7, Corollaries 10 and 11, and Propositions 12 and 14 imply that the conditions of Theorem 20 are satisfied and therefore 2) is satisfied. Obviously 2) implies 1). As we consider the correspondence $\mathcal{P}_{min}$, Corollary 4 implies that we have inheritance of superadditivity. Then by Theorem 3, 1) is equivalent to 3) and 4). □

5.3 Equivalence between cycle-complete condition for G and adjacent cycle condition for G'.

We have established in Section 4 that the Myerson's game on a graph G corresponds to a restriction of the $\mathcal{P}_{\min}$-restricted game for a specific weighted graph G'. We now prove that the Path, Star, Cycle, and Pan conditions are always satisfied on G', and that the Adjacent cycles condition is satisfied on G' if and only if G is cycle-complete.

Let $\gamma = \{1, e_1, 2, e_2, \ldots, m, e_m, m+1\}$ be an elementary path in G'. If $w_1 = w_m = \frac{1}{2}$, then e_1 and e_m are incident to s. We necessarily have $2 = s$ and $m = 2$ otherwise γ would form a cycle. Then γ trivially satisfies Path condition. Otherwise we can assume w.l.o.g. $w_1 = 1$ and as $w_i = 1$ or $\frac{1}{2}$ for all i, $1 \leq i \leq m$, we trivially have $w_i \leq \max(w_1, w_m) = 1$ for all i, $1 \leq i \leq m$. Hence Path condition is satisfied.

Let us now consider a star $\{e_1, e_2, e_3\}$ with $e_1 = \{1,2\}$, $e_2 = \{1,3\}$, and $e_3 = \{1,4\}$. We can assume $w_1 \leq w_2 \leq w_3$. As $w(e) = \frac{1}{2}$ if and only if e is incident to s we only have three possible cases represented in Figures 18. Therefore Star condition is obviously satisfied.

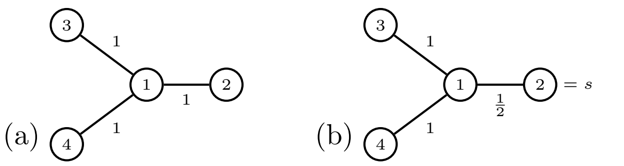

Figure 18: (a) : $w_1 = w_2 = w_3 = 1$, (b) : $w_1 = \frac{1}{2} < 1 = w_2 = w_3$, (c) : $w_1 = w_2 = w_3 = \frac{1}{2}$.

Let us now consider a simple cycle $C_m = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$. If $s \notin V(C_m)$ then $w_1 = w_2 = \cdots = w_m = 1$. Otherwise we can assume w.l.o.g. $s = 2$ and then $w_1 = w_2 = \frac{1}{2} < 1 = w_3 = w_4 = \cdots = w_m$ as represented in Figure 19. Hence Cycle condition is also satisfied.

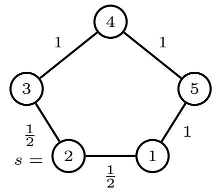

Figure 19: $w_1 = w_2 = \frac{1}{2} < 1 = w_3 = \cdots = w_m$.

Let us consider a simple cycle $C_m = \{e_1, e_2, \ldots, e_m\}$ and an elementary path P_r such that $|V(C_m) \cap V(P_r)| = 1$. If $s \notin V(C_m)$ then $w_1 = w_2 = \cdots = w_m = 1$. Otherwise $\min_{1 \leq k \leq m} w_k = \frac{1}{2}$. We have $w(e) \in \{\frac{1}{2}, 1\}$ for all $e \in E$. Therefore there is no edge e in P_r with $w(e) < \min_{1 \leq k \leq m} w_k$. Hence Pan condition is satisfied (in fact this condition is not relevant).

Proposition 22. *Cycle complete condition for G is equivalent to Adjacent cycle condition for G'.*

Proof. By contradiction, let us assume there is a cycle $C_m = \{1, e_1, 2, e_2, \ldots, e_m, 1\}$ in G which is not complete. After renumbering if necessary we can assume that $\{1,j\} \notin E$ with $j \in \{3,\dots,m-1\}$ and j as small as possible. If $j \geq 4$, then $\tilde{e} = \{1, j-1\}$ is a chord of C_m. We can replace C by the smaller cycle $\tilde{C} = \{1, \tilde{e}, j-1, e_{j-1}, j, e_j, \dots, m, e_m, 1\}$ (which is not complete) as represented in Figure 20. Hence we can assume $j = 3$. We can assume

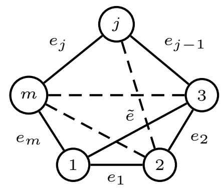

Figure 20: $\{1,j\} \notin E(C)$ with $j \geq 4$.

that there is no chord $\{k,l\}$, with $k \neq 2$ and $l \neq 2$, otherwise we can replace C by a smaller cycle. Hence any chord of C is incident to 2. Let us now consider the two adjacent cycles $\tilde{C}$ and $\tilde{C}^{'}$ in $G^{'}$, obtained adding the edges $\tilde{e}_1 = \{s,1\}$ and $\tilde{e}_3 = \{s,3\}$, $\tilde{C} = \{s, \tilde{e}_1, 1, e_1, 2, e_2, 3, \tilde{e}_3, s\}$, and $\tilde{C}^{'} = \{s, \tilde{e}_3, 3, e_3, 4, \dots, m, e_m, 1, \tilde{e}_1, s\}$, as represented in Figure 21. Then

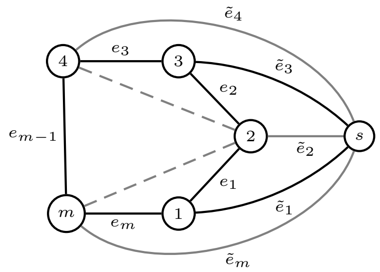

Figure 21: $\tilde{C}$ and $\tilde{C}^{'}$ in $G^{'}$.

$\tilde{C}$ and $\tilde{C}^{'}$ have no maximum weight chord. $\tilde{C}$ has only one chord $\{s,2\}$ of weight $\frac{1}{2}$. We have $|\tilde{C}| = 4$ and $|\tilde{C}'| \geq 4$, and $\tilde{e}_1$, $\tilde{e}_3$ are two common non-maximum weight edges of $\tilde{C}$ and $\tilde{C}^{'}$. Hence $\tilde{C}$ and $\tilde{C}^{'}$ contradict the Adjacent cycles condition in $G^{'}$.

Conversely assume that the Adjacent cycles condition is not satisfied in $G^{'}$. Let $\tilde{C}$ and $\tilde{C}^{'}$ be two adjacent cycles in $G^{'}$ satisfying the conditions of the Adjacent cycles condition except that they have two common non-maximum weight edges $\tilde{e}_1$ and $\tilde{e}_3$ (of weight $\frac{1}{2}$) and $|V(\tilde{C})|$, $|V(\tilde{C}^{'})| \geq 4$. Then $\tilde{e}_1$ and $\tilde{e}_3$ are necessary incident to s which is a common vertex of $\tilde{C}$ and $\tilde{C}^{'}$. By assumption $\tilde{C}$ and $\tilde{C}^{'}$ have no maximum weight chord (of weight 1), and no common chord. Moreover $\tilde{C}$ can have at most one non-maximum weight chord. As s is linked to all the other vertices we necessarily have $|V(\tilde{C})| = 4$ (otherwise $\tilde{C}$ has more than one chord). We set $\tilde{C} = \{s, \tilde{e}_1, 1, e_1, 2, e_2, 3, \tilde{e}_3, s\}$, and $\tilde{C}^{'} = \{s, \tilde{e}_3, 3, e_3, 4, \dots, m, e_m, 1, \tilde{e}_1, s\}$, as represented in Figure 21. Then we consider the cycle $C = \{1, e_1, 2, e_2, 3, e_3, \dots, m, e_m, 1\}$ $(= (\tilde{C}^{'} \setminus \{s\}) \cup \{2\})$. We have $|V(C)| \geq 4$. Then a chord of C

can only be incident to 2 (otherwise it is a chord of $\tilde{C}'$ of maximum weight, contradicting the assumption). Therefore C is a non complete cycle and G is not cycle-complete. Let us now observe that the second part of the Adjacent cycles condition is always satisfied. Let $\tilde{C}$ and $\tilde{C}'$ be two adjacent cycles in G' satisfying the conditions of the Adjacent cycles condition and with one common non-maximum weight edge e_1. Then e_1 is incident to s and therefore s is a common vertex of $\tilde{C}$ and $\tilde{C}'$. As all edges incident to s have weight $\frac{1}{2}$, we always have $w_1 = w_2 = w_2' = \frac{1}{2}$ (with the notations of the Adjacent cycles condition). $\square$

It is interesting to observe that to establish the inheritance of convexity from (N, v) to $(N, \overline{v})$ with G' we only need to verify the Adjacent cycles condition. Moreover to verify the Adjacent cycles condition it is sufficient in this case to verify the non existence of a pair of adjacent cycles C and C' with $|V(C)| = 4$ and $|V(C')| \geq 4$ and having two common non-maximum weight edges (and satisfying the other conditions of the Adjacent cycles condition).

5.4 A description of graphs satisfying the necessary conditions for inheritance of $\mathcal{F}$-convexity

We now describe more precisely the connected graphs satisfying the necessary conditions defined in Section 5.1.

Proposition 23. *Let $G = (N, E, w)$ be a weighted connected graph. Let E_1 be the set of minimum weight edges in E and N_1 be the set of their end-vertices in N. If the edge-weight function w satisfies the Star, Path, Cycle, Pan, and Adjacent cycles conditions then:*

1) If an elementary path γ in G has its first edge in E_1 then the edge-weights of γ are non decreasing.

2) $G_1 = (N_1, E_1)$ is a connected subgraph.

3) If $|E_1| = 1$ then there exists at most one chordless cycle $\tilde{C}$ with $E(\tilde{C}) \cap E_1 \neq \emptyset$. For all cycle $C_m = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$ with $m \geq 3$ and $E(C_m) \cap E_1 = \emptyset$, either C_m has constant edge-weights or $\sigma(N) < w_1 = w_2 < w_3 = \cdots = w_m = M$, where $M = \max_{e \in E(C_m)} w(e)$. In this last case $\{1, 3\}$ is a maximum weight chord of C_m and 2 is a cut vertex[4] of G.

4) If $|E_1| \geq 2$ then for all cycle $C_m = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$ with $m \geq 3$ we have $\sigma(N) \leq w_1 = w_2 \leq w_3 = \cdots = w_m = M$. If $\sigma(N) < w_1 = w_2 < M$, then $\{1, 3\}$ is a maximum weight chord and 2 is a cut vertex of G.

[4] A *cut vertex* (or articulation point) in a graph is a vertex the removal of which disconnects the graph.

If $\sigma(N) = w_1 = w_2 < M$ and if there exists an edge in E_1 non incident to 2, then 2 is a cut vertex of G (but $\{1, 3\}$ is not necessarily a chord).

We give an example of a graph satisfying Proposition 23 with $|E_1| = 1$ (resp. $|E_1| \geq 2$) in Figure 22 (resp. Figure 23).

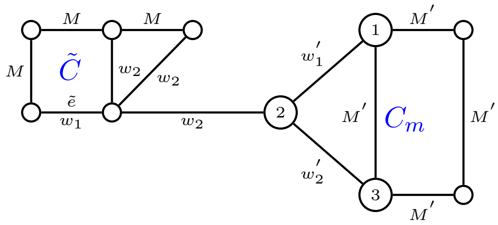

Figure 22: $E_1 = \{\tilde{e}\}$, $w(\tilde{e}) = w_1 < w_2 \leq M$ in $\tilde{C}$ and $w_1' = w_2' < M'$ in C_m.

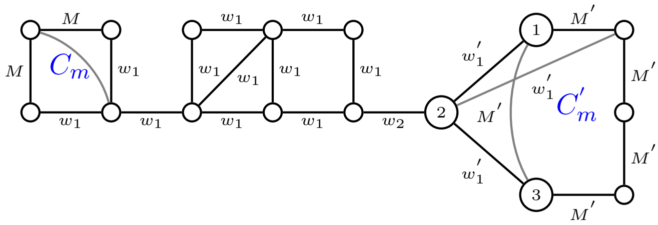

Figure 23: $|E_1| \geq 2$, $\sigma(N) = w_1 < M$, and $w_1 \leq w_2 < w_1' = w_2' < M'$.

Proof. 1) Immediately results from Path condition.

2) Let us consider v' and v'' in N_1. By definition v' and v'' are end-vertices of edges e' and e'' in E_1 such that $w(e') = w(e'') = \sigma(N)$. If $e' = e''$ or if e' and e'' are adjacent then $e' \cup e''$ corresponds to a path in G_1 linking v' to v''. Otherwise let γ be a shortest path in G linking e' to e''. Then Path condition applied to $\gamma' = e' \cup \gamma \cup e''$ implies $w(e) \leq \max(w(e'), w(e'')) = \sigma(N)$ and therefore $w(e) = \sigma(N)$ for all edge $e \in \gamma$. Hence γ' is a path from v' to v'' in G_1.

3) Let $\tilde{e}$ be the unique edge in E_1. By contradiction, let us assume that $\tilde{e}$ is a common edge of two cycles C and C' without chords. Then the Adjacent cycles condition implies that some edge of C or C' adjacent to $\tilde{e}$ has a weight equal to $w(\tilde{e})$ and therefore $|E_1| \geq 2$, a contradiction. For a cycle C_m such that $\tilde{e} \notin E(C_m)$, the Pan condition implies the result. In particular every path γ linking $\tilde{e}$ to C_m has to end at vertex 2. Therefore if we delete vertex 2 the graph is disconnected.

4) Cycle condition applied to $C_m = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$ implies $w_1 \leq w_2 \leq w_3 = \cdots = w_m = M$, after renumbering if necessary,

and that all chords of C_m have weight w_2 or M. If $\{e_1, e_2\} \not\subseteq E_1$ then there necessarily exists an edge $e \in E_1 \setminus E(C_m)$ and Pan condition implies either $\sigma(N) < w_1 = w_2 = M$ or $\sigma(N) < w_1 = w_2 < M$. In this last case $\{1,3\}$ is a maximum weight chord of C_m and 2 is an articulation point. If $\{e_1, e_2\} \subseteq E_1$ then $\sigma(N) = w_1 = w_2 \leq M$. Let us assume $\sigma(N) = w_1 = w_2 < M$ and that there exists $e \in E_1$ non incident to 2. No path γ can link e to some vertex $j \in V(C_m) \setminus \{2\}$, otherwise it contradicts Path condition applied to the path linking e to e_1 or e_2 passing through vertex j and edges of maximum weight of C_m as represented in Figure 24. 2 is not necessarily a cut vertex if all edges of E_1 are incident to it as it is shown in Figure 25.

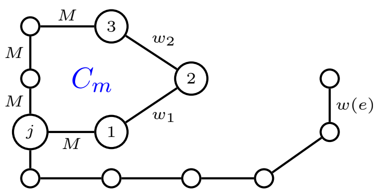

Figure 24: $|E_1| \geq 2$, $\sigma(N) = w_1 = w_2 = w(e) < M$.

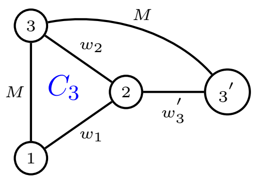

Figure 25: $E_1 = \{e_1, e_2, e_3'\}$, $\sigma(N) = w_1 = w_2 = w_3' < M$.

$\square$

6 Conclusion

Our main result gives necessary and sufficient conditions for inheritance of $\mathcal{F}$-convexity with the $\mathcal{P}_{\min}$ correspondence. Although $\mathcal{F}$-convexity is a weaker condition than convexity this result presents interesting aspects. First $\mathcal{F}$-convexity is in itself an interesting property as connected subsets correspond to natural coalitions and as incentive to cooperate is more natural between connected players. Moreover we can establish that the conditions for inheritance of $\mathcal{F}$-convexity can be checked in polynomial time but this result goes beyond the scope of the paper. Secondly the necessary conditions are also valid for inheritance of convexity. In a forthcoming work we will present supplementary necessary conditions to have inheritance of convexity. These conditions are so restrictive that the edge-weights can have only three different values. Therefore $\mathcal{F}$-convexity can constitute a good alternative to convexity as the class of graphs satisfying the inheritance of $\mathcal{F}$-convexity is much larger than the one satisfying inheritance of convexity. This work also constitutes a first step for other correspondences. In particular the correspondence $\mathcal{P}_G$ associated with the strength of a graph presented in [10], which gives natural partitions, coincides with $\mathcal{P}_{\min}$ on cycle-free graphs. Hence the Star and Path conditions restricted to induced stars and paths are also valid for this correspondence and a natural extension of this work could be to find parallel results for $\mathcal{P}_G$. Moreover the inheritance of superadditivity for $\mathcal{P}_G$ is not always satisfied and its characterization is not obvious. We could also consider another restricted game $(N, \tilde{v})$ defined by $\tilde{v}(A) = \sum_{l=1}^{p} \overline{v}(A_l)$ for all $A \subseteq N$, where $A_1, A_2, \ldots, A_p$ are the connected components of A. It can be shown that for the $\mathcal{P}_{\min}$ correspondence, the Star and Path conditions are valid for both games. Hence we can address the problem of the extension of Theorem 20 to $(N, \tilde{v})$, but it seems that additional conditions are required. Borm, Owen and Tijs [6] introduced arc games. It is well known that if the communication graph is cycle-free then there is inheritance of convexity from the underlying game to the corresponding arc game [14]. We can also define restricted arc games, by substituting to the partition into connected components the correspondence $\mathcal{P}_{\min}$. The question of inheritance of convexity is more difficult as we do not even necessarily have inheritance of superadditivity.

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